SOME RATIONAL VERTEX ALGEBRAS

Dražen Adamović

Department of Mathematics, University of Zagreb, Zagreb, Croatia

Abstract. Let $L((n-\frac{3}{2})\Lambda_0)$, $n \in \mathbb{N}$, be a vertex operator algebra associated to the irreducible highest weight module $L((n-\frac{3}{2})\Lambda_0)$ for a symplectic affine Lie algebra. We find a complete set of irreducible modules for $L((n-\frac{3}{2})\Lambda_0)$ and show that every module for $L((n-\frac{3}{2})\Lambda_0)$ from the category $\mathcal{O}$ is completely reducible.

0. Introduction

Let $\mathfrak{g}$ be a type one affine Lie algebra. Then the irreducible highest weight $\mathfrak{g}$–module $L(k\Lambda_0)$ has a natural vertex operator algebra structure for every $k \in \mathbb{C}$, $k \neq -g$. When k is a positive integer, then the vertex operator algebra $L(k\Lambda_0)$ is rational and its irreducible modules are integrable highest weight modules of level k (cf. [DL], [MP], [FZ]).

In this paper we will consider the case of a symplectic affine Lie algebra of the type $C_\ell^{(1)}$ and the corresponding vertex operator algebra $L((n-\frac{3}{2})\Lambda_0)$, $n \in \mathbb{N}$. We give the description of two sets of admissible weights S_1^n and S_2^n and prove that $L(\lambda)$, $\lambda \in S_1^n \cup S_2^n$, are irreducible $L((n-\frac{3}{2})\Lambda_0)$–modules (cf. Section 2 and 3). Next, we prove that irreducible $L((n-\frac{3}{2})\Lambda_0)$–modules are in one-to-one correspondence with zeros of the set of polynomials $\mathcal{P}_{0,\ell}$ (Section 4). By using this correspondence we show that the set $\{L(\lambda) \mid \lambda \in S_1^n \cup S_2^n\}$ gives a complete list of irreducible $L((n-\frac{3}{2})\Lambda_0)$–modules. The classification of irreducible $L((n-\frac{3}{2})\Lambda_0)$–modules implies that every $L((n-\frac{3}{2})\Lambda_0)$–module from the category $\mathcal{O}$ is completely reducible (cf. Section 6).

It turns out that representations of the vertex operator algebra $L((n-\frac{3}{2})\Lambda_0)$ are in some respects quite "similar" to the integrable highest weight representations.

The author expresses the thanks to M.Primc for suggesting to him the study of this problem and for helpful discussions and valuble comments.

1. Symplectic affine algebra

The symplectic affine (Kac-Moody) Lie algebra $C_\ell^{(1)}$ can be written as

$$\mathfrak{g} = sp_{2\ell}(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c + \mathbb{C}d$$

1991 *Mathematics Subject Classification.* 17B67.

Key words and phrases. Symplectic affine Lie algebra, Vertex operator algebra, Admissible weights, Modules for vertex operator algebra, Rational vertex operator algebra.

with the usual commutation relations (cf. [K]). For $X \in sp_{2\ell}(\mathbb{C})$ and $n \in \mathbb{Z}$ we write $X(n) = X \otimes t^n$. Consider two ℓ-dimensional vector spaces $A_1 = \sum_{i=1}^{\ell} \mathbb{C} a_i$, $A_2 = \sum_{i=1}^{\ell} \mathbb{C} a_i^*$. Let $A = A_1 + A_2$. The Weyl algebra $W(A)$ is the associative algebra over $\mathbb{C}$ generated by A and relations

$$[a_i, a_j] = [a_i^*, a_j^*] = 0, \quad [a_i, a_i^*] = \delta_{i,j}, \qquad i, j \in \{1, 2, \ldots, \ell\}.$$

Define the normal ordering on A by

$$:xy: = \tfrac{1}{2}(xy + yx) \qquad x, y \in A.$$

Then (cf. [B] and [FF]) all such elements $: xy :$ span a Lie algebra isomorphic to $\overset{\circ}{\mathfrak{g}} = sp_{2\ell}(\mathbb{C})$ with a Cartan subalgebra $\overset{\circ}{\mathfrak{h}}$ spanned by

$$h_i = - : a_i a_i^* : \quad i = 1, 2, ..., \ell.$$

Let $\{\epsilon_i \mid 1 \leqslant i \leqslant \ell\} \subset \overset{\circ}{\mathfrak{h}}{}^*$ be the dual basis such that $\epsilon_i(h_j) = \delta_{i,j}$. The root system of $\overset{\circ}{\mathfrak{g}}$ is given by

$$\Delta = \{\pm(\epsilon_i \pm \epsilon_j), \pm 2\epsilon_i \mid 1 \leqslant i, j \leqslant \ell, i < j\}$$

with $\alpha_1 = \epsilon_1 - \epsilon_2, ..., \alpha_{\ell-1} = \epsilon_{\ell-1} - \epsilon_\ell$, $\alpha_\ell = 2\epsilon_\ell$ being a set of simple roots. The highest root is $\theta = 2\epsilon_1$. Let $\overset{\circ}{\mathfrak{g}} = \overset{\circ}{\mathfrak{n}}_- + \overset{\circ}{\mathfrak{h}} + \overset{\circ}{\mathfrak{n}}_+$ be the corresponding triangular decomposition. We fix the root vectors :

$$X_{\epsilon_i - \epsilon_j} = :a_i a_j^*:, \quad X_{\epsilon_i + \epsilon_j} = :a_i a_j:, \quad X_{-(\epsilon_i + \epsilon_j)} = :a_i^* a_j^*: \ .$$

2. Some admissible weights

Let R (resp R_+)$\subset \mathfrak{h}$ be the set of real (resp positive real) coroots of $\mathfrak{g}$. Fix $\lambda \in \mathfrak{h}^*$. Let $R^\lambda = \{\alpha \in R \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}\}$, $R_+^\lambda = R^\lambda \cap R_+$, Π the set of simple coroots in R and $\Pi^\lambda = \{\alpha \in R_+^\lambda \mid \alpha$ not equal to a sum of several roots from $R_+^\lambda\}$. Define ρ in the usual way.

Recall that a weight $\lambda \in \mathfrak{h}^*$ is called admissible (cf. [KW 2]) if the following properties are satisfied :

(1) $\langle \lambda + \rho, \alpha \rangle \notin -\mathbb{Z}_+$ for all $\alpha \in R_+$,
(2) $\mathbb{Q}R^\lambda = \mathbb{Q}\Pi$.

Let $M(\lambda)$ denote the Verma module with the highest weight λ, $M^1(\lambda)$ its maximal submodule and $L(\lambda)$ its irreducible quotient.

First let us recall some results of V.Kac and M.Wakimoto that we shall use:

Theorem 1. *(Kac – Wakimoto, Cor. 2.1. in* [KW 1] *) Let λ be an admissible weight. Then*

$$L(\lambda) = \frac{M(\lambda)}{\sum_{\alpha \in \Pi^\lambda} U(\mathfrak{g}) v^\alpha} \ ,$$

where $v^\alpha \in M(\lambda)$ is a singular vector of weight $r_\alpha.\lambda$, the highest weight vector of $M(r_\alpha.\lambda) = U(\mathfrak{g}) v^\alpha \subset M(\lambda)$.

Theorem 2. *(Kac–Wakimoto, Theorem 4.1 in* [KW 2]*) Let V be a $\mathfrak{g}$– modul from the category $\mathcal{O}$ such that for any irreducible subquotient $L(\mu)$ the weight μ is admissible. Then $\mathfrak{g}$–modul V is completely reducible.*

Denote by P_+ the set of all dominant integral weights.

Theorem 3. *(Kac–Wakimoto, Cor.4.1 in* [KW 2]*) Let $\Lambda \in P_+$ and λ be an admissible weight. Then the $\mathfrak{g}$ –modul $L(\Lambda) \otimes L(\lambda)$ decomposes into a dirtect sum of irreducible $\mathfrak{g}$– modules $L(\mu)$ with μ admissible highest weight and $R^{\mu} = R^{\lambda}$.*

Let $\Pi = \{\alpha_0^\vee, \dots, \alpha_\ell^\vee\}$, $c = \alpha_0^\vee + \dots + \alpha_\ell^\vee$ and set

$$\Pi_1 = \{2c - (h_1 + h_2), h_1 - h_2, \dots, h_{\ell-1} - h_\ell, h_\ell\} = \{2\alpha_0^\vee + \alpha_1^\vee, \alpha_1^\vee, \dots, \alpha_\ell^\vee\},$$

$$\Pi_2 = \{c - h_1, h_1 - h_2, \dots, h_{\ell-1} - h_\ell, h_{\ell-1} + h_\ell\} = \{\alpha_0^\vee, \dots, \alpha_{\ell-1}^\vee, \alpha_{\ell-1}^\vee + 2\alpha_\ell^\vee\}.$$

Let S_i denote the set of all admissible λ with $\Pi^\lambda = \Pi_i,\ i = 1, 2$.

Lemma 4. *Let $\lambda \in S_i$, $i = 1, 2$. Then*

$$\langle \lambda, c\rangle = n - \tfrac{3}{2} \quad \textit{for some } n \in \mathbb{N}\ .$$

Proof. For $\lambda \in S_1$ we have

$$\langle \lambda + \rho, 2\alpha_0^\vee + \alpha_1^\vee\rangle = \langle \lambda, 2\alpha_0^\vee + \alpha_1^\vee\rangle + 3$$
$$= \langle \lambda, 2\alpha_0^\vee + 2\alpha_1^\vee + \cdots 2\alpha_\ell^\vee\rangle - \langle \lambda, \alpha_0^\vee + 2\alpha_1^\vee + \cdots 2\alpha_\ell^\vee\rangle + 3 > 0.$$

This implies $\langle \lambda, c\rangle > -\frac{3}{2}$ and we see that $\langle \lambda, c\rangle \in -\frac{3}{2} + \mathbb{N}$. Similary we prove the case i=2. □

Let

$$S_i^n = \{\lambda \in S_i \mid \langle \lambda, c\rangle = n - \tfrac{3}{2}\} \quad i = 1, 2,$$
$$P_+^1 = \{\lambda \in P_+ \mid \langle \lambda, c\rangle = 1\} = \{\Lambda_0, \dots, \Lambda_\ell\}.$$

Then $S_i = \cup_{n\in\mathbb{N}}\ S_i^n$. We give a description of S_1^n and S_2^n for $n \in \mathbb{N}$:

Proposition 5.

(1)
$$S_1^1 = \{-\tfrac{1}{2}\Lambda_0, -\tfrac{3}{2}\Lambda_0 + \Lambda_1\}\ ,$$
$$S_1^{n+1} = \{S_1^n + P_+^1\} \cup \{-(n + \tfrac{3}{2})\Lambda_0 + (2n + 1)\Lambda_1\}, \quad n \in \mathbb{N};$$

(2)
$$S_2^1 = \{-\tfrac{1}{2}\Lambda_\ell, -\tfrac{3}{2}\Lambda_\ell + \Lambda_{\ell-1}\}\ ,$$
$$S_2^{n+1} = \{S_2^n + P_+^1\} \cup \{-(n + \tfrac{3}{2})\Lambda_\ell + (2n + 1)\Lambda_{\ell-1}\}\ , \quad n \in \mathbb{N}.$$

Proof. We can directly obtain the description of the set S_1^1.

By the definition of sets S_i^n we have

$$\{S_1^n + P_+^1\} \subset S_1^{n+1} \text{ and } (n + \tfrac{3}{2})\Lambda_0 + (2n + 1)\Lambda_1 \in S_1^{n+1}.$$

Let $\lambda \in S_1^{n+1}$, $\lambda \neq -(n + \frac{3}{2})\Lambda_0 + (2n + 1)\Lambda_1$. Then $\langle \lambda, \alpha_0^\vee\rangle = -(n - m + \frac{1}{2})$, for $m \in \mathbb{Z}_+$. Since $\langle \lambda + \rho, 2\alpha_0^\vee + \alpha_1^\vee\rangle > 0$ we have $\langle \lambda, \alpha_1^\vee\rangle \geqslant (2(n - m) - 1)$, and this implies

$$\lambda = -(n - m + \tfrac{1}{2})\Lambda_0 + (2(n - m) - 1)\Lambda_1 + \Lambda^{(1)} + \cdots + \Lambda^{(m+1)}$$

where $\Lambda^{(i)} \in P_+^1, \quad i = 1, \dots, m + 1$. We have obtained

$$\lambda \in S_1^{(n-m)} + P_+^1 + \cdots + P_+^1 \subset S_1^n + P_+^1$$

and (1) holds.

The proof of (2) is similar. □

3. Modules for Vertex operator algebra $L((n-\frac{3}{2})\Lambda_0)$

We know that the generalized Verma module $N(k\Lambda_0)$ with the highest weight $k\Lambda_0$, $k \in \mathbb{C}$, is a vertex operator algebra if $k \neq -g$ (here g denotes the dual Coxeter number). The irreducible quotient $L(k\Lambda_0)$ of $N(k\Lambda_0)$ is also a vertex operator algebra (see [FLM], [FgF],[DL], [FZ] and [MP]).

As usual we shall denote by $Y(w,z) = \sum_{m\in\mathbb{Z}} w_m z^{-m-1}$ the vertex operator (or the field) of the vector w.

Let V be a $\mathfrak{g}$–module of level k, $k \neq -g$ from the category $\mathcal{O}$ (or a highest weight module) and let

$$X(z) = Y(X(-1)\mathbf{1}, z) = \sum_{m\in\mathbb{Z}} X(m) z^{-m-1}, \quad X \in \mathring{\mathfrak{g}},$$

be the family of fields acting on V defined by the action of $X(m) \in \mathfrak{g}$. By Theorem 4.3 in [MP] or Theorem 2.4.1 in [FZ] there is a unique extension of these fields that make V into a module over the vertex operator algebra $N(k\Lambda_0)$. Hence we may identify $\mathfrak{g}$–modules of level k in the category $\mathcal{O}$ with the $N(k\Lambda_0)$–modules in the category $\mathcal{O}$.

Moreover, if I is an ideal of the vertex operator algebra $N(k\Lambda_0)$, then a $\mathfrak{g}$–module from the category $\mathcal{O}$ is a module of the vertex operator algebra $N(k\Lambda_0)/I$ if and only if $Y(w,z)V = 0$ for all $w \in I$ (or equivalently, for all generators w of the ideal I) (cf. Corrollary 3.2 and Proposition 4.2 below).

We will find all irreducible representations of the vertex operator algebras $L((n-\frac{3}{2})\Lambda_0)$, $n \in \mathbb{N}$, associated to the symplectic algebra $C_\ell^{(1)}$.

Put $\lambda_n = (n-\frac{3}{2})\Lambda_0$. Then λ_n is an admissible weight with $\Pi^{\lambda_n} = \Pi_1$.

Put $\gamma_0 = \delta - (\epsilon_1 + \epsilon_2)$. It is easy to show that $\gamma_0^\vee = 2\alpha_0^\vee + \alpha_1^\vee$. Then we have:

$$r_{\gamma_0}.\lambda_n = \lambda_n - 2n\gamma_0, \quad r_{\alpha_i}.\lambda_n = \lambda_n - \alpha_i, \quad i = 1,2,...,\ell.$$

By $\mathbf{1}$ we denote a highest weight vector in $N(\lambda_n)$.

Theorem 1. *The maximal submodule of* $N(\lambda_n)$ *is* $N^1(\lambda_n) = U(\mathfrak{g})v_n$*, where*

$$v_n = (X_{\epsilon_1+\epsilon_2}(-1)^2 - X_{2\epsilon_1}(-1)X_{2\epsilon_2}(-1))^n\mathbf{1}, \quad n \in \mathbb{N}.$$

Proof. It can be checked by a direct calculation that v_n is a singular vector of weight $\lambda_n - 2n\gamma_0$. Since

$$v^{\alpha_i} = X_{-\alpha_i}(0)\mathbf{1} = 0$$

for $i = 1,2,...,\ell$, we conclude from Theorem 2.1 that v_n generates the maximal submodule $N^1(\lambda_n)$. $\square$

Clearly we have

$$Y(v_n, z) = (X_{\epsilon_1+\epsilon_2}(z)^2 - X_{2\epsilon_1}(z)\,X_{2\epsilon_2}(z))^n.$$

Theorem 1 implies the following :

Corollary 2. *Let V be $\mathfrak{g}$–module from the category $\mathcal{O}$ of level $n-\frac{3}{2}$. Then*
$$(X_{\epsilon_1+\epsilon_2}(z)^2 - X_{2\epsilon_1}(z)\,X_{2\epsilon_2}(z))^n = 0 \quad on\ V$$
if and only if V is $L(\lambda_n)$–module.

A.Feingold and I.Frenkel gave the bosonic construction (see [FF]) of four irreducible $\mathfrak{g}$–modules of level $-\frac{1}{2}$: $L(\mu_1), L(\mu_2), L(\mu_3), L(\mu_4)$ where
$$\mu_1 = -\tfrac{1}{2}\Lambda_0,\ \mu_2 = -\tfrac{3}{2}\Lambda_0 + \Lambda_1,\ \mu_3 = -\tfrac{1}{2}\Lambda_\ell,\ \mu_4 = -\tfrac{3}{2}\Lambda_\ell + \Lambda_{\ell-1}.$$
By using Lemma 7 in [FF] and the explicit construction (Theorem A in [FF]) we obtain:

Proposition 3. *On $L(\mu_i)$, $\quad i = 1, 2, 3, 4$, we have*
$$X_{\epsilon_1+\epsilon_2}(z)^2 - X_{2\epsilon_1}(z)\,X_{2\epsilon_2}(z) = 0\ .$$

Theorem 4. *Let $V(n-\frac{3}{2})$ be an irreducible $L(\lambda_n)$–module and $V(1)$ an irreducible $L(\Lambda_0)$–module. Then*
$$V(n-\tfrac{3}{2}) \otimes V(1)$$
is a $L(\lambda_{n+1})$–module.

Proof. By Theorem 2.2 vector $\mathbf{1} \otimes \mathbf{1} \in L(\lambda_n) \otimes L(\Lambda_0)$ generates the submodule isomorphic to $L(\lambda_{n+1})$. It is easy to show that $L(\lambda_{n+1})$ is a subalgebra of the vertex operator algebra $L(\lambda_n) \otimes L(\Lambda_0)$ in the sense of [FZ]. Since $V(n-\frac{3}{2}) \otimes V(1)$ is a module for $L(\lambda_n) \otimes L(\Lambda_0)$ (cf. Proposition 10.1 in [DL]) it is also a module for $L(\lambda_{n+1})$. $\square$

Remark. *The Theorem 4 can also be proved by using Corrolary 2 and the vertex operator formula for integrable highest weight representations (cf.* [LP]*, Proposition 5.5).*

Lemma 5. *Let $\lambda \in S_1^n \cup S_2^n$. Then $L(\lambda)$ is a $L(\lambda_n)$–modul.*

Proof. Induction on $n \in \mathbb{N}$. For $n = 1$ we have $S_1^1 \cup S_2^1 = \{\mu_1, \mu_2, \mu_3, \mu_4\}$. Then $L(\mu_i)$, $i = 1, 2, 3, 4$ are $L(-\frac{1}{2}\Lambda_0)$ modules by Proposition 3.

First notice that $L(\Lambda)$ for $\Lambda \in P_+^1$ is a $L(\Lambda_0)$–module (cf. [FZ], [DL], [MP]). Assume that $L(\lambda')$ is a $L(\lambda_n)$–modul for all $\lambda' \in S_1^n \cup S_2^n$. Let $\lambda \in S_1^{n+1} \cup S_2^{n+1}$. If $\lambda = \lambda_0 + \Lambda$, $\lambda_0 \in S_1^n \cup S_2^n$, $\Lambda \in P_+^1$, then $L(\lambda_0)\ \otimes L(\Lambda)$ is a $L(\lambda_{n+1})$–module by Theorem 4. Since $v_{\lambda_0} \otimes v_\Lambda$ is a singular vector of weight $\lambda_0 + \Lambda$, by Theorem 2.2 it generates the submodule isomorphic to $L(\lambda)$ and $L(\lambda)$ is a $L(\lambda_{n+1})$–module.

Let $\lambda = -(n+\frac{3}{2})\Lambda_0 + (2n+1)\Lambda_1$. Put $\mu = -(n+\frac{1}{2})\Lambda_0 + (2n-1)\Lambda_1$. Then $L(\mu) \otimes L(\Lambda_0)$ is a $L(\lambda_{n+1})$–module. Since
$$v = 2X_{2\epsilon_1}(-1)v_\mu \otimes v_{\Lambda_0} + (2n+1)v_\mu \otimes X_{2\epsilon_1}(-1)v_{\Lambda_0}$$
is a singular vector in $L(\mu) \otimes L(\Lambda_0)$ of weight $\lambda - \delta$, it generates the submodule isomorphic to $L(\lambda)$.

Let $\lambda = -(n+\frac{3}{2})\Lambda_\ell + (2n+1)\Lambda_{\ell-1}$. Put $\nu = -(n+\frac{1}{2})\Lambda_\ell + (2n-1)\Lambda_{\ell-1}$. Then $L(\nu) \otimes L(\Lambda_\ell)$ is a $L(\lambda_{n+1})$–modul. Since
$$v_1 = 2X_{-2\epsilon_\ell}(0)v_\nu \otimes v_{\Lambda_\ell} + (2n+1)v_\nu \otimes X_{-2\epsilon_\ell}(0)v_{\Lambda_\ell}$$
is a singular vector in $L(\nu) \otimes L(\Lambda_\ell)$, it generates the submodule isomorphic to $L(\lambda)$. $\square$

Remark. *It follows from Lemma 5 that $L(\lambda)$, $\lambda \in S_1^n \cup S_2^n$, is an irreducible $L(\lambda_n)$–module. In what follows we prove that these are all irreducible $L(\lambda_n)$–modules (cf. Lemma 6.1 and Theorem 6.2).*

4. Classification of irreducible representations

Fix $n \in \mathbb{N}$. For $w \in U(\overset{\circ}{\mathfrak{g}})v_n$ and $j \in \mathbb{Z}$ put $w(j) = w_{j+2n-1}$. Then $w(j)$ has operator degree j (i.e. $[d, w(j)] = jw(j)$).

Set

$$\overline{W} = \coprod_{j\in\mathbb{Z}} W(j), \quad W(j) = \mathbb{C}-span\ \{w(j)|\ w \in U(\overset{\circ}{\mathfrak{g}})v_n\ \}.$$

By using the commutator formula for vertex operators we get (cf. [MP]) the following:

Proposition 1. *$\overline{W}$ is a loop module under the adjoint action of $\mathfrak{g}$. In particular,*

$$[X(i), w(j)] = (X.w)(i + j)$$

for $X \in \overset{\circ}{\mathfrak{g}}$, $w \in U(\overset{\circ}{\mathfrak{g}})v_n$, $i, j \in \mathbb{Z}$.

Then $W(0)$ is a finite dimensional $\overset{\circ}{\mathfrak{g}}$–module with the highest weight $2n(\epsilon_1+\epsilon_2) = 2n\ \omega_2$. By $W(0)_0$ denote the zero-weight subspace of $W(0)$.

Proposition 2. *Let V be an irreducible highest weight module of level $n-\frac{3}{2}$ with the highest weight vector v_λ. The following statements are equivalent :*

1. *V is a $L(\lambda_n)$–module;*
2. *$\overline{W}V = 0$;*
3. *$W(0)_0 v_\lambda = 0$.*

Proof.

The equivalence of (1) and (2) was already discussed in the introduction of Section 3.

Clearly (2) implies (3).

For the converse first notice that by assumption $V = M(\lambda)/M^1(\lambda)$. Hence to see (2) it is enough to see $\overline{W}M(\lambda) \subset M^1(\lambda)$, i.e. $\overline{W}M(\lambda) \neq M(\lambda)$ (since $\overline{W}M(\lambda)$ is a submodule, and $M^1(\lambda)$ is the maximal submodule).

Since $\overline{W}M(\lambda) = \overline{W}U(\mathfrak{n}_-)v_\lambda = U(\mathfrak{n}_-)\overline{W}v_\lambda$, we have

$$\overline{W}M(\lambda) \neq M(\lambda) \quad \text{iff} \quad v_\lambda \in \overline{W}M(\lambda) \quad \text{iff} \quad W(0)_0 v_\lambda = 0. \quad \square$$

Let $u \in W(0)_0$. Clearly there exists the uniqe polynomial $p_u \in S(\overset{\circ}{\mathfrak{h}})$ such that

$$uv_\lambda \ = \ p_u(\lambda)v_\lambda.$$

Set $\mathcal{P}_{0,\ell} = \{\ p_u \mid u \in W(0)_0\}$. We have

Corollary 3. *There is one-to-one correspondence between :*

1. *irreducible $L(\lambda_n)$–modules from the category $\mathcal{O}$;*
2. *$\lambda \in \mathfrak{h}^*$ such that $p(\lambda) = 0$ for all $p \in \mathcal{P}_{0,\ell}$.*

5. Zeros of some polynomials

Denote by $_L$ the adjoint action of $\mathring{\mathfrak{g}}$ on $U(\mathring{\mathfrak{g}})$: $X_L f = [X, f]$ for $X \in \mathring{\mathfrak{g}}$ and $f \in U(\mathring{\mathfrak{g}})$. The following lemma is obtained by direct calculations:

Lemma 1.

(1) $(X^k_{2\epsilon_1})_L(X^n_{-2\epsilon_1}) \in X^{n-k}_{-2\epsilon_1}(-1)^k 4^k n\cdots(n-k+1)\cdot(h_1-n+k)\cdots(h_1-n+1) + U(\mathring{\mathfrak{g}})\mathring{\mathfrak{n}}_+$,
(2) $(X^n_{2\epsilon_1})_L(X^n_{-2\epsilon_1}) \in (-1)^n 4^n n!\cdot h_1\cdots(h_1-n+1) + U(\mathring{\mathfrak{g}})\mathring{\mathfrak{n}}_+$
(3) $(X^n_{2\epsilon_2})_L(X^n_{-2\epsilon_2}) \in (-1)^n 4^n n!\cdot h_2\cdots(h_2-n+1) + U(\mathring{\mathfrak{g}})\mathring{\mathfrak{n}}_+$,
(4) $(X^m_{\epsilon_1+\epsilon_2})_L(X^m_{-\epsilon_1-\epsilon_2}) \in (-1)^m m!\cdot(h_1+h_2)\cdots(h_1+h_2-m+1) + U(\mathring{\mathfrak{g}})\mathring{\mathfrak{n}}_+$,
(5) $(X^{m'}_{\epsilon_1+\epsilon_2})_L(X^m_{-\epsilon_1-\epsilon_2}) \in U(\mathring{\mathfrak{g}})X_{\epsilon_1+\epsilon_2}$ *for* $m' > m$,
(6) $(X^r_{\epsilon_1+\epsilon_2})_L(X^k_{-2\epsilon_2}) \in U(\mathring{\mathfrak{g}})\mathring{\mathfrak{n}}_+$ *for* $r > 0$,
(7) $(X^{2k}_{\epsilon_1+\epsilon_2})_L(X^k_{-2\epsilon_1}) = (2k)!X^k_{2\epsilon_2}$,
(8) $(X^{2k+i}_{\epsilon_1+\epsilon_2})_L(X^k_{-2\epsilon_1}) = 0$ *for* $i > 0$,
(9) $p(h)X^k_\alpha = X^k_\alpha p(h + k\alpha(h))$ *for any polynomial* p.

Lemma 2. *Let*

$$f = X_{\beta_1}\cdots X_{\beta_k},\quad X_{\beta_i}\in\mathring{\mathfrak{n}}_+,\quad [X_{\beta_i}, X_{\beta_j}] = 0,\quad \textit{for all}\quad i,j;$$
$$g = X_{\gamma_1}\cdots X_{\gamma_m},\quad X_{\gamma_i}\in\mathring{\mathfrak{n}}_-,\quad [X_{\gamma_i}, X_{\gamma_j}] = 0,\quad \textit{for all}\quad i,j;$$
$$\sum_{i=1}^k \beta_i + \sum_{i=1}^m \gamma_i = 0.$$

Then

(1) $f_L g \in X_{\beta_1}\cdots X_{\beta_k}X_{\gamma_1}\cdots X_{\gamma_m} + U(\mathring{\mathfrak{g}})\mathring{\mathfrak{n}}_+$,
(2) $g_L f \in (-1)^m X_{\beta_1}\cdots X_{\beta_k}X_{\gamma_1}\cdots X_{\gamma_m} + U(\mathring{\mathfrak{g}})\mathring{\mathfrak{n}}_+$.

We shall also use the following consequence of the binomial formula :

Lemma 3. *For a polynomial* q *of degree* $\deg(q) < n$ *we have*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} q(k) = 0.$$

In this Section we consider the case C_2 and calculate some polynomials from $\mathcal{P}_{0,2}$.

Lemma 4. *Let:*

(1) $p_1(h) = (h_1-h_2)(h_1-h_2-1)\ldots(h_1-h_2-2n+1)$;
(2) $p_2(h) = (h_1-n+\frac{3}{2})(h_1-n+\frac{5}{2})\ldots(h_1+\frac{1}{2})h_2(h_2-1)\ldots(h_2-n+1)$;
(3) $p_3(h) = \sum_{k=0}^n \frac{n!4^n}{k!4^k}(h_1+h_2-2n+1)\ldots(h_1+h_2-2n+2k)h_2(h_2-1)\ldots(h_2-n+k+1)$.

Then $p_1, p_2, p_3 \in \mathcal{P}_{0,2}$.

Proof.

We identify $\mathring{\mathfrak{g}} \otimes t^0$ with $\mathring{\mathfrak{g}}$ and write X instead of $X(0)$. Clearly for $a_1, a_2 \ldots, a_r \in \mathring{\mathfrak{g}}$ we have

$$\begin{aligned}
&[\mathrm{Coeff}_{z^{-2n}} \quad (a_1 \cdot a_2 \cdots a_r)_L (X_{\epsilon_1+\epsilon_2}(z)^2 - X_{2\epsilon_1}(z) X_{2\epsilon_2}(z))^n] v_\lambda \\
&\quad = [(a_1 \cdot a_2 \cdots a_r)_L (X_{\epsilon_1+\epsilon_2}(0)^2 - X_{2\epsilon_1}(0) X_{2\epsilon_2}(0))^n] v_\lambda .
\end{aligned}$$

Hence

$$W(0) v_\lambda = W v_\lambda, \qquad W(0)_0 v_\lambda = W_0 v_\lambda,$$

where

$$W = U(\mathring{\mathfrak{g}})_L (X^2_{\epsilon_1+\epsilon_2} - X_{2\epsilon_1} X_{2\epsilon_2})^n \subset U(\mathring{\mathfrak{g}})$$

and where W_0 denotes the zero weight subspace of W.

(1) First notice that

$$(X^{2n}_{\epsilon_1-\epsilon_2} X^{2n}_{-2\epsilon_1})_L (X^2_{\epsilon_1+\epsilon_2} - X_{2\epsilon_1} X_{2\epsilon_2})^n = 8^n (X^{2n}_{\epsilon_1-\epsilon_2})_L (X^2_{-\epsilon_1+\epsilon_2} - X_{-2\epsilon_1} X_{2\epsilon_2})^n \in W_0$$

We have

$$\begin{aligned}
&(X^{2n}_{\epsilon_1-\epsilon_2})_L (X^2_{-\epsilon_1+\epsilon_2} - X_{-2\epsilon_1} X_{2\epsilon_2})^n \\
&= (X^{2n}_{\epsilon_1-\epsilon_2})_L X^{2n}_{-\epsilon_1+\epsilon_2} + \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k (X^{2n}_{\epsilon_1-\epsilon_2})_L X^{2k}_{-\epsilon_1+\epsilon_2} X^{n-k}_{-2\epsilon_1} X^{n-k}_{2\epsilon_2} \\
&\in C p_1(h) + U(\mathring{\mathfrak{g}}) \mathring{\mathfrak{n}}_+
\end{aligned}$$

for some constant $C \neq 0$.

(2) First notice that

$$(X^n_{-2\epsilon_1} X^n_{-2\epsilon_2})_L (X^2_{\epsilon_1+\epsilon_2} - X_{2\epsilon_1} X_{2\epsilon_2})^n \in W_0 .$$

By Lemma 2 we may calculate the corresponding polynomial from

$$\begin{aligned}
&(X^2_{\epsilon_1+\epsilon_2} - X_{2\epsilon_1} X_{2\epsilon_2})^n_L (X^n_{-2\epsilon_1} X^n_{-2\epsilon_2}) \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (X^k_{2\epsilon_2} X^{2n-2k}_{\epsilon_1+\epsilon_2} X^k_{2\epsilon_1})_L (X^n_{-2\epsilon_1} X^n_{-2\epsilon_2}).
\end{aligned}$$

By using Lemma 1 we have:

$$\begin{aligned}
&(X^k_{2\epsilon_1})_L X^n_{-2\epsilon_1} X^n_{-2\epsilon_2} \in X^{n-k}_{-2\epsilon_1} X^n_{-2\epsilon_2} \\
&\cdot (-1)^k 4^k n(n-1) \cdots (n-k+1)(h_1 - n + k) \cdots (h_1 - n + 1) + U(\mathring{\mathfrak{g}}) \mathring{\mathfrak{n}}_+
\end{aligned}$$

and

$$\begin{aligned}
&(X^{2n-2k}_{\epsilon_1+\epsilon_2})_L (X^{n-k}_{-2\epsilon_1} X^n_{-2\epsilon_2}) = [(X^{2n-2k}_{\epsilon_1+\epsilon_2})_L (X^{n-k}_{-2\epsilon_1}] X^n_{-2\epsilon_2} + \\
&\sum_{i=1}^{2n-2k} \binom{2n-2k}{i} [(X^{2n-2k-i}_{\epsilon_1+\epsilon_2})_L (X^{n-k}_{-2\epsilon_1}] [(X^i_{\epsilon_1+\epsilon_2})_L X^n_{-2\epsilon_2}] \\
&\in [(X^{2n-2k}_{\epsilon_1+\epsilon_2})_L X^{n-k}_{-2\epsilon_1}] X^n_{-2\epsilon_2} + U(\mathring{\mathfrak{g}}) \mathring{\mathfrak{n}}_+ \quad (\text{ by Lemma 1.6}) \\
&\quad (2n-2k)!\, X^{n-k} X^{n} + U(\mathring{\mathfrak{g}}) \mathring{\mathfrak{n}}_+ \quad (\text{by Lemma 1.7})
\end{aligned}$$

We have obtained

$$(X_{2\epsilon_2}^k X_{\epsilon_1+\epsilon_2}^{2n-2k} X_{2\epsilon_1}^k)_L X_{-2\epsilon_1}^n X_{-2\epsilon_2}^n$$
$$\in X_{2\epsilon_2}^n X_{-2\epsilon_2}^n (-1)^k 4^k n(n-1)\cdots(n-k+1)$$
$$\cdot(2n-2k)!(h_1-n+k)\cdots(h_1-n+1)+U(\mathring{\mathfrak{g}})\mathring{\mathfrak{n}}_+$$
$$=(-1)^n n! 4^n h_2\cdots(h_2-n+1)$$
$$\cdot\sum_{k=0}^{n}\binom{n}{k}(2n-2k)!4^k n\cdots(n-k+1)(h_1-n+k)\cdots(h_1-n+1)+U(\mathring{\mathfrak{g}})\mathring{\mathfrak{n}}_+.$$

For $h_1=-\frac{1}{2}+j$, $j=0,1,\ldots,n$ we can show

$$\sum_{k=0}^{n}\binom{n}{k}(2n-2k)!4^k n\cdots(n-k+1)(h_1-n+1)\cdots(h_1-n+k)$$

$$=(2n)!!\sum_{k=0}^{n}\binom{n}{k}(2n-2k-1)!!(-2n+2+2j-1)\cdots(-2n+2k+2j-1)$$
$$=(2n)!!(2n-2j-1)!!\sum_{k=0}^{n}(-1)^k\binom{n}{k}(2n-2k-1)\cdots(2n+1-2j-2k)$$
$$=0 \qquad \text{(by using Lemma 3).}$$

This implies

$$\sum_{k=0}^{n}\binom{n}{k}(2n-2k)!4^k n\cdots(n-k+1)(h_1-n+1)\cdots(h_1-n+k)$$
$$=4^n n!(h_1-n+\tfrac{3}{2})(h_1-n+\tfrac{5}{2})\ldots(h_1+\tfrac{1}{2})$$

and we have

$$(X_{-2\epsilon_1}^n X_{-2\epsilon_2}^n)_L (X_{\epsilon_1+\epsilon_2}^2 - X_{2\epsilon_1}X_{2\epsilon_2})^n \in Cp_2(h)+U(\mathring{\mathfrak{g}})\mathring{\mathfrak{n}}_+$$

for some constant $C\neq 0$.

(3) First notice that

$$(X_{\epsilon_1+\epsilon_2}^{2n})_L (X_{-\epsilon_1-\epsilon_2}^2 - X_{-2\epsilon_1}X_{-2\epsilon_2})^n \in W_0.$$

By using Lemma 1 we can show

$$(X_{\epsilon_1+\epsilon_2}^{2n})_L (X_{-2\epsilon_1}^k X_{-\epsilon_1-\epsilon_2}^{2n-2k} X_{-2\epsilon_2}^k)$$
$$\in (2n)!4^k k!(-1)^k h_2\cdot(h_2-k+1)(h_1+h_2-2n+1)\cdots(h_1+h_2-2k)+U(\mathring{\mathfrak{g}})\mathring{\mathfrak{n}}_+.$$

By using this and Lemma 1 we see that

$$(X_{\epsilon_1+\epsilon_2}^{2n})_L (X_{-\epsilon_1-\epsilon_2}^2 - X_{-2\epsilon_1}X_{-2\epsilon_2})^n \in (2n)!p_3(h)+U(\mathring{\mathfrak{g}})\mathring{\mathfrak{n}}_+. \quad \square$$

The following lemma describes the set

$$T^n=\{h\in\mathbb{C}^2 \mid p_1(h)=p_2(h)=p_3(h)=0\}.$$

Lemma 5. *$T^n = T_1^n \cup T_2^n$, where*

$$\begin{aligned} T_1^n &= \{(s+2r,,s) \mid s = 0,1,\dots,n-r-1\ ,\ r = 0,1,\dots,n-1\} \\ &\cup \quad \{(s+2r+1,s) \mid s = 0,1,\dots,n-r-1\ ,\ r = 0,1,\dots,n-1\}, \\ T_2^n &= \{(s+2r,s) \mid s = -r-\tfrac{1}{2},\dots,n-2r-\tfrac{3}{2},\ r = 0,1,\dots,n-1\} \\ &\cup \quad \{(s+2r+1,s) \mid s = -r-\tfrac{3}{2},\dots,n-2r-\tfrac{5}{2},\ r = 0,1,\dots,n-1\}. \end{aligned}$$

Proof. Fix $n \in \mathbb{N}$ and let $T_{1,2} = \{h \in \mathbb{C}^2 \ \mid p_1(h) = p_2(h) = 0\}$. Then $T_{1,2} = T_{1,2}^1 \cup T_{1,2}^2$ where

$$T_{1,2}^1 = \{(k,k') \in \mathbb{Z}^2 \mid k' = 0,\dots,n-1, k = k',\dots,k'+2n-1\},$$

$$T_{1,2}^2 = \{(k,k') \in (Z+\tfrac{1}{2})^2 \mid k = -\tfrac{1}{2}+i,\ i = 0,\dots,n-1;\ k' = k-j,\ j = 0,\dots,2n-1\}.$$

Clearly $h \in T^n$ if and only if $p_3(h) = 0$.

Let $(h_1,h_2) \in T_{1,2}$ and $h_1 - h_2 = 2r, \quad r = 0,\dots,n-1$. Put $h_2 = s$. Then for $\tilde{p_3}(s) = p_3(s+2r,s)$ we have

$$\begin{aligned} \tilde{p_3}(s) &= \sum_{k=0}^{n} \frac{n!4^n}{k!4^k}(2s-2n+2r+1)\cdots(2s-2n+2r+2k)\cdot\cdots\ (s-n+k+1) \\ &= 4^n(n-r)!r!\binom{s}{n-r}\cdot\sum_{k=0}^{n}\binom{s-n+r+k}{r}\binom{s-n+r+k-\frac{1}{2}}{k}. \end{aligned}$$

Let $(2r+s,s) \in T_{1,2}$. Clearly $(2r+s,s) \in T^n$ if and only if $\tilde{p_3}(s) = 0$. It is easy to see that

$$\tilde{p_3}(s) = 0 \text{ for } s = 0,\dots,n-r-1 \quad \text{and} \quad \tilde{p_3}(s) \neq 0 \text{ for } s = n-r,\dots,n-1.$$

Let $s = -r-\frac{1}{2}+i, \quad i = 0,\dots,n-r-1$. Then we have

$$\begin{aligned} \frac{1}{4^n(n-r)!r!}\tilde{p_3}(s) &= \binom{s}{n-r}\sum_{k=0}^{n}\binom{-n-\frac{1}{2}+i+k}{r}\binom{-n-1+i+k}{k} \\ &= \binom{s}{n-r}\sum_{k=0}^{n}(-1)^k\binom{n-i}{k}\binom{-n-\frac{1}{2}+i+k}{r} = 0 \end{aligned}$$

by using Lemma 3.

Let $s = -r-\frac{1}{2}-i, \quad i = 1,\dots,r$. Then we have (by using Lemma 3)

$$\frac{1}{\binom{s}{n-r}4^n(n-r)!r!}\tilde{p_3}(s) = \sum_{k=0}^{n}(-1)^k\binom{n+i}{k}\binom{-n-\frac{1}{2}-i+k}{r}$$

$$\begin{aligned} &= \sum_{k=0}^{n+i}(-1)^k\binom{n+i}{k}\binom{-n-\frac{1}{2}-i+k}{r} - \sum_{k=n+1}^{n+i}(-1)^k\binom{n+i}{k}\binom{-n-\frac{1}{2}-i+k}{r} \\ &= -\sum_{k=n+1}^{n+i}(-1)^k\binom{n+i}{k}\binom{-n-\frac{1}{2}-i+k}{r} \\ &= (-1)^{n+i+r+1}\sum_{k=0}^{i-1}(-1)^k\binom{n+i}{k}\binom{k+r-\frac{1}{2}}{r}. \end{aligned}$$

Since
$$\frac{\binom{n+i}{k+1}\binom{k+r+\frac{1}{2}}{r}}{\binom{n+i}{k}\binom{k+r-\frac{1}{2}}{r}} = \frac{(n+i-k)(k+r+\frac{1}{2})}{(k+1)(k+\frac{1}{2})} > 1$$
we can easily show that $\tilde{p_3}(s) \neq 0$.

Similarly we treat the case
$$(h_1, h_2) \in T_{1,2}, \quad h_1 - h_2 = 2r+1, \quad r = 0, \ldots, n-1$$
and obtain the result. $\square$

It follows from Lemma 5 :

Lemma 6.

(1) $T_1^{n+1} = T_1^n \cup \{T_1^n + (1,0)\} \cup \{T_1^n + (1,1)\} \cup \{(2n+1, 0)\}$;
(2) $T_2^{n+1} = T_2^n \cup \{T_2^n + (1,0)\} \cup \{T_2^n + (1,1)\} \cup \{(n - \frac{1}{2}, -n - \frac{3}{2})\}$.

6. The main result

Lemma 1. *Let $L(\lambda)$ be a $L(\lambda_n)$–module. Then $\lambda \in S_1^n \cup S_2^n$.*

Proof. Let $L(\lambda)$ be a $L(\lambda_n)$–module. Since
$$(X_{\epsilon_j+\epsilon_{j+1}}(0)^2 - X_{2\epsilon_j}(0)X_{2\epsilon_{j+1}}(0))^n \quad \in W, \quad j = 1, \ldots, \ell$$
we can use results for the case C_2 and obtain that
$$(\lambda(h_j), \lambda(h_{j+1})) \in T^n, \quad 1 \leqslant j \leqslant \ell.$$
Let $\tilde{S}_i^n = \{\lambda \in \mathfrak{h}^* | \ \langle \lambda, c\rangle = n - \frac{3}{2}, \ \ (\lambda(h_j), \lambda(h_{j+1})) \in T_i^n, \quad 1 \leqslant j \leqslant \ell\}$, i=1,2. We will prove by induction that $\tilde{S}_i^n = S_i^n$ for all $n \in \mathbb{N}$, $i = 1, 2$.

For $n = 1$ we have $T_1^1 = \{(0,0), (1,0)\}$ and $T_2^1 = \{(-\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, -\frac{3}{2})\}$. Then for $\lambda \in \tilde{S}_1^1 \cup \tilde{S}_2^1$ we get
$$(\lambda(h_1), \ldots, \lambda(h_\ell)) \in \{(0, \ldots, 0), (1, 0, \ldots, 0), (-\tfrac{1}{2}, \ldots, -\tfrac{1}{2}), (-\tfrac{1}{2}, \ldots, -\tfrac{1}{2}, -\tfrac{3}{2})\}$$
which implies
$$\tilde{S}_1^1 = \{-\tfrac{1}{2}\Lambda_0, -\tfrac{3}{2}\Lambda_0 + \Lambda_1\} = S_1^1,$$
$$\tilde{S}_2^1 = \{-\tfrac{1}{2}\Lambda_\ell, -\tfrac{3}{2}\Lambda_\ell + \Lambda_{\ell-1}\} = S_2^1.$$

Assume that $\tilde{S}_i^n = S_i^n$. Let $\lambda \in \tilde{S}_1^{n+1}$.

If $(\lambda(h_1), \lambda(h_2)) = (2n+1, 0)$ then $\lambda = -(n + \frac{3}{2})\Lambda_0 + (2n+1)\Lambda_1$.

If $(\lambda(h_1), \lambda(h_2)) \neq (2n+1, 0)$ then $(\lambda(h_j), \lambda(h_{j+1})) \in T_1^n \cup \{T_1^n + (1,0)\} \cup \{T_1^n + (1,1)\}$ $j = 1, \ldots, \ell$.

We define $\Lambda \in \mathfrak{h}^*$ by
$$\langle \Lambda, h_j\rangle = \begin{cases} 0 & \text{if} \quad (\lambda(h_j), \lambda(h_{j+1}) \in T_1^n \\ 1 & \text{otherwise} \end{cases} \qquad \text{for } j = 1, \ldots, \ell - 1,$$
$$\langle \Lambda, h_\ell\rangle = \begin{cases} 0 & \text{if} \quad (\lambda(h_{\ell-1}), \lambda(h_\ell) \in \{T_1^n + (1,0)\} \cup T_1^n \\ 1 & \text{otherwise} \end{cases},$$
$$\langle \Lambda, c\rangle = 1$$

Let $\lambda' = \lambda - \Lambda$. It is easy to show that $\Lambda \in P^1_+$ and $\lambda' \in \tilde{S}^n_1$. Since $\{\tilde{S}^n_1 + P^1_+\} \subset \tilde{S}^{n+1}_1$ we have obtained

$$\begin{aligned}\tilde{S}^{n+1}_1 &= \{\tilde{S}^n_1 + P^1_+\} \cup \{-(n+\tfrac{3}{2})\Lambda_0 + (2n+1)\Lambda_1\} \\ &= \{S^n_1 + P^1_+\} \cup \{-(n+\tfrac{3}{2})\Lambda_0 + (2n+1)\Lambda_1\} = S^{n+1}_1\end{aligned}$$

(by using Proposition 2.5) .

Similary we prove

$$\tilde{S}^{n+1}_2 = \{\tilde{S}^n_2 + P^1_+\} \cup \{-(n+\tfrac{3}{2})\Lambda_\ell + (2n+1)\Lambda_{\ell-1}\} = S^{n+1}_2$$

and we conclude by induction that $\tilde{S}^n_i = S^n_i$, $i = 1, 2$. $\square$

Theorem 2.

(1) *The set $\{L(\lambda) \mid \lambda \in S^n_1 \cup S^n_2\}$ provides a complete list of irreducible $L(\lambda_n)$–modules.*
(2) *Let V be a $L(\lambda_n)$–module from the category $\mathcal{O}$. Then V decomposes into a direct sum of irreducible $L(\lambda_n)$–modules.*

Proof. (1) By using Lemma 3.5 and Lemma 1 we have that $L(\lambda_n)$–modules are exactly $L(\lambda)$ for $\lambda \in S^n_1 \cup S^n_2$.

(2) Let $L(\mu)$ be an irreducible subquotient of V. Then $L(\mu)$ is a $L(\lambda_n)$–module and by Lemma 1 we have that $\mu \in S^n_1 \cup S^n_2$. By using Theorem 2.2 we obtain that V is completely reducible. $\square$

Remark. *In* [Z] *and* [FZ] *are defined representations of vertex operator algebras which need not be in category $\mathcal{O}$. Vertex operator algebra is by definition rational if it has only finitely many irreducible modules and if every finitely generated module is a direct sum of irreducible ones. By the abuse of language (or by changing the definition) one could say that Theorem 2 states that the vertex operator algebra $L((n-\frac{3}{2})\Lambda_0)$, $n \in \mathbb{N}$, is rational.*

By using Theorem 2 we obtain:

Corollary 3. *Let V be a highest weight $\mathfrak{g}$–module of level $n - \frac{3}{2}$. The following statements are equivalent :*

(1) *V is an irreducible $L(\lambda_n)$–module;*
(2) $\overline{W}V = 0$.

Corollary 4. *Let $n \in \mathbb{N}$ and $\langle \lambda, c\rangle = n - \frac{3}{2}$. We have*

$$\overline{W}M(\lambda) = \begin{cases} M^1(\lambda) & \textit{for all } \lambda \in S^n_1 \cup S^n_2, \\ M(\lambda) & \textit{otherwise.}\end{cases}$$

References

[B] Bourbaki, *Groupes et algèbras de Lie*, Hermann, Paris, 1975.

[DL] C.Y.Dong and J.Lepowsky, *Generalized vertex algebras and relative vertex operators*, Birkhäuser, Baston, 1993.

[FgF] B.Feigin and E.Frenkel, *Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras*, Proceedings of the RIMS Research Project 1991, Infinite Analysis A **7** (1992), 197-215.

[FF] A.J.Feingold and I.B.Frenkel, *Classical affine algebras*, Adv. in Math. **56** (1985), 117-172.

[FLM] I.B. Frenkel, J. Lepowsky and A.Meurman, *Vertex Operator Algebras and the Monster*, Pure and Applied Math., Academic Press, San Diego, 1988.

[FZ] I.B.Frenkel and Y.Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras*, Duke Math. Journal **66** (1992), 123-168.

[K] V.G.Kac, *Infinite dimensional Lie algebras*, Birkhäuser, Basel, 1983.

[KW 1] V.G.Kac and M.Wakimoto, *Modular invariant representations of infinite dimensional Lie algebras and superalgebras*, Proc. Natl. Acad. Sci. USA **85** (1988), 4956-4960.

[KW 2] V.Kac and M.Wakimoto, *Classification of modular invariant representations of affine algebras*, in Infinite Dimensional Lie algebras and groups, Advanced Series in Math. Phys. **7** (1989), World Scientific.

[LP] J.Lepowsky and M.Primc, *Structure of the standard modules for affine Lie algebra* $A_1^{(1)}$, Contemporary Math., AMS, Providence **46** (1985).

[MP] A.Meurman and M.Primc, *Annihilating fields for* $sl_2(\mathbb{C})\tilde{}$ *and combinatorial identities (manuscript)*.

[Z] Y.Zhu, *Vertex operator algebras, eliptic function and modular forms* (1990), Ph. D. dissertation, Yale Univ.

Department of Mathematics, University of Zagreb, Bijenička 30, 41000 Zagreb, Croatia

E-mail address: `adamovic@cromath.math.hr`